ON THE RELATIVISTIC VLASOV–POISSON SYSTEM

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Abstract

The Cauchy problem is revisited for the so-called relativistic Vlasov–Poisson system in the attractive case, originally studied by Glassey and Schaeffer in 1985. It is proved that a unique global classical solution exists whenever the positive, integrable initial datum f_0 is spherically symmetric, compactly supported in momentum space, vanishes on characteristics with vanishing angular momentum, and its $\mathfrak{L}^\beta$ norm is below a critical constant $C_\beta > 0$ whenever $\beta \geq 3/2$. It is also shown that, if the bound C_β on the $\mathfrak{L}^\beta$ norm of f_0 is replaced by a bound $C > C_\beta$, any $\beta \in (1, \infty)$, then classical initial data exist which lead to a blow-up in finite time. The sharp value of C_β is computed for all $\beta \in (1, 3/2]$, with the results $C_\beta = 0$ for $\beta \in (1, 3/2)$ and $C_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$ (when $\|f_0\|_{\mathfrak{L}^1} = 1$), while for all $\beta > 3/2$ upper and lower bounds on C_β are given which coincide as $\beta \downarrow 3/2$. Thus, the $\mathfrak{L}^{3/2}$ bound is optimal in the sense that it cannot be weakened to an $\mathfrak{L}^\beta$ bound with $\beta < 3/2$, whatever that bound. A new, non-gravitational physical vindication of the model which (unlike the gravitational one) is not restricted to weak fields, is also given.

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1 Introduction

In [GlSc85], Glassey and Schaeffer inaugurated a series of studies (see [GlSc01] and the references therein; see also [HaRe07]) of what they sanctioned the "relativistic Vlasov–Poisson system" (rVP in the following). The rVP poses a classical Cauchy problem for a relative density function $f_t : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ of an N-body system with Cauchy data $f_0 \in (\mathfrak{P} \cap \mathfrak{C}^1)(\mathrm{d}p\mathrm{d}q)$,[1] given in form of the kinetic equation

$$\Big(\partial_t + v \cdot \nabla_q + \sigma \nabla_q \phi_t(q) \cdot \nabla_p\Big) f_t(p, q) = 0, \tag{1}$$

in which the velocity $v \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$ of a (point) particle of unit mass are related by Einstein's formula (with the speed of light $c = 1$),

$$v = \frac{p}{\sqrt{1 + |p|^2}}\,; \tag{2}$$

the scalar field $\phi_t : \mathbb{R}^3 \to \mathbb{R}_-$ satisfies the Poisson equation

$$\Delta_q \phi_t(q) = 4\pi \int_{\mathbb{R}^3} f_t(p, q)\mathrm{d}p \tag{3}$$

with asymptotic condition[2]

$$\phi_t(q) \asymp -|q|^{-1} \tag{4}$$

when $|q| \to \infty$, so that[3] $\phi_t = -|\mathrm{id}\,|^{-1} * \int f_t \mathrm{d}p$; and $\sigma \in \{-1, +1\}$ decides whether the gradient force field of ϕ is attractive ($\sigma = -1$) or repulsive ($\sigma = +1$). In this paper we are primarily interested in the attractive case $\sigma = -1$.

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[1] By $\mathfrak{P}(\mathrm{d}p\mathrm{d}q)$ we denote the probability measures, by $\mathfrak{P}_n(\mathrm{d}p\mathrm{d}q)$ those having finite n-th moment, and by $(\mathfrak{P}_n \cap \mathfrak{L}^\alpha)(\mathrm{d}p\mathrm{d}q)$, $\alpha \geq 1$, respectively $(\mathfrak{P}_n \cap \mathfrak{C}^1)(\mathrm{d}p\mathrm{d}q)$, those of these measures which are absolutely continuous w.r.t. Lebesgue measure $\mathrm{d}p\mathrm{d}q$ on $\mathbb{R}^3 \times \mathbb{R}^3$ (momentum×physical space) with density in $\mathfrak{L}^\alpha(\mathrm{d}p\mathrm{d}q)$, respectively in $\mathfrak{C}^1(\mathrm{d}p\mathrm{d}q)$ which are the functions with one continuous classical derivative (abusing notation, we identify these measures with their densities). While the relative density function f_t thus fulfills the requirements of a probability density function, it should really be thought of as a continuum approximation to a *merely normalized* (i.e. relative) empirical "density" on (p, q)-space of an actual individual N-body system. Incidentally, the conspicuous absence of N in (1)–(4) means that we study the evolution on suitable time and space scales. The scaling transformation $t \mapsto N^{-1}t$, $q \mapsto N^{-1}q$, so that $v \mapsto v$ and $p \mapsto p$, together with $f \mapsto N^3 f$ and $\phi \mapsto \phi$ restores N explicitly in (1),(3),(4). Note that $(\mathfrak{P} \cap \mathfrak{L}^1)(\mathrm{d}p\mathrm{d}q)$ is invariant under this scaling map.

[2] In principle, asymptotic conditions other than (4) can be imposed, for instance other harmonic behavior indicating "system-external sources at infinity."

[3] Note that $\phi_t = -|\mathrm{id}\,|^{-1} * \int f_t \mathrm{d}p$ does *not* represent dynamical degrees of freedom beyond those of f_t. Thus, we will speak of solutions f_t of (1)–(4).

The rVP system is not truly relativistic in the sense of proper Lorentz or even general covariance. Yet for the mathematically special (and physically idealized) situation of spherical systems, rVP can actually be obtained from truly relativistic (and physically relevant) Vlasov models in certain limiting physical regimes; hence, rVP may have some physical significance. Its version with $\sigma = +1$ (denoted rVP$^+$) is obtained directly from the relativistic Vlasov–Maxwell system (rVM) for a single specie of electrically charged physical particles with spherically symmetric initial data without any further conditions; cf. [Hor90]. Thus, in the repulsive case $\phi_t(q)$ can be thought of as Coulomb's electrical potential at the space point q at time t. The version with $\sigma = -1$ (denoted rVP$^-$) ought to obtain in a "weak field limit" of the physically relevant general covariant Vlasov–Einstein system (VE) with spherically symmetric data, though we are only aware of some work (see [Ren94]) on the combined weak field plus low velocity limit which leads to the familiar Vlasov–Poisson system with $\sigma = -1$ (VP$^-$), formally obtained from (1)–(4) by replacing Einstein's formula (2) with Newton's $v = p$. Thus, in the attractive case $\phi_t(q)$ may be thought of as Newton's gravitational potential at the space point q at time t. Unfortunately, this gravitational interpretation has to be taken with a grain of salt, for some mathematically interesting phenomena such as stationary bound states [Bat89, HaRe07] and finite-time blow up [GlSc85] (signaling gravitational collapse to a singularity, see [LMR08b]) of this (both psychologically and "physically") attractive version of rVP occur in the *strong field* regime, i.e. were rVP$^-$ can no longer be expected to be a legitimate approximation to VE. While this would seem to make mathematical studies of the strong field regime of rVP$^-$ questionable from a physical perspective, in the appendix we give an unconventional (and presumably surprising) physical interpretation of rVP$^-$ with spherical symmetry in terms of distributional solutions of rVM for a neutral two-species plasma which is not restricted to weak fields, and which could have interesting applications in space physics; thus the mathematically rigorous vindication of this electrical interpretation of rVP$^-$ is an important open problem.

In the main part of the present paper we revisit the questions of global existence and uniqueness versus finite time blow-up of solutions to rVP$^-$, which were addressed already by Glassey and Schaeffer [GlSc85]. We restrict our discussion to classical solutions, but ask for the optimal — in the sense of weakest — constraints that guarantee that classical data will launch a unique global solution of the dynamical system. In this vein, we prove the following result about rVP$^-$:

Theorem 1.1. *A unique global classical solution of rVP$^-$ exists for all spherically symmetric initial data $f_0 \in \mathfrak{P}_1 \cap \mathfrak{C}^1(\mathrm{d}p\mathrm{d}q)$ which are compactly supported in momentum space, vanish for $p \times q = 0$, and satisfy $\|f_0\|_{3/2} < C_{3/2}$, with $C_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3} \approx 0.367$. The $\mathfrak{L}^{3/2}$ bound $C_{3/2}$ on f_0 is optimal in the sense that initial data f_0 exist which satisfy all the hypotheses except that $\|f_0\|_{3/2} > C_{3/2}$, and which lead to a blow-up in finite time.*

Remark 1.2. *The critical case $\|f_0\|_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$ is not covered by our theorem.*

Remark 1.3. *All global-in-time solutions covered by Theorem 1.1 have positive, those that blow up in finite time non-positive energy. Among the data that lead to finite time blow-up there are indeed some with zero energy. This improves on Glassey–Schaeffer's result that negative energy data will lead to finite time blow-up.*

Remark 1.4. *By the interpolation inequality, $f_0 \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta$ with $\beta > 3/2$ implies $f_0 \in P_1 \cap \mathfrak{L}^{3/2}$, with $\|f_0\|_\beta < C_{3/2}^{3(1-1/\beta)}$ implying $\|f_0\|_{3/2} < C_{3/2}$. This shows that a global existence and uniqueness theorem analogous to Theorem 1.1 can also be stated with the sharp $\mathfrak{L}^{3/2}$ condition on f_0 replaced by a sharp $\mathfrak{L}^\beta$ condition on f_0 for any $\beta > 3/2$, with $C_{3/2}$ replaced by a corresponding sharp constant $C_\beta \geq C_{3/2}^{3(1-1/\beta)}$. Beside the sharp $C_{3/2}$ given in Theorem 1.1, and the lower bound on C_β for $\beta > 3/2$ just stated, we will also give an explicit upper bound on C_β for $\beta > 3/2$.*

Remark 1.5. *The reverse to the interpolation estimates of course is not true: $f_0 \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$ with $\|f_0\|_{3/2} < C_{3/2}$ does not imply any bound on $\|f_0\|_\beta$ for $\beta > 3/2$. Thus, our $\mathfrak{L}^{3/2}$ condition is weaker than any of the possible $\mathfrak{L}^\beta$ conditions with $\beta > 3/2$. In fact, our $\mathfrak{L}^{3/2}$ bound is the weakest possible $\mathfrak{L}^\beta$ condition for which an analog of Theorem 1.1 can be formulated, in the sense that our $\mathfrak{L}^{3/2}$ bound on f_0 cannot be replaced by an $\mathfrak{L}^\beta$ bound with $\beta \in (1, 3/2)$, whatever that bound. Indeed, among the data f_0 satisfying any such $\mathfrak{L}^\beta$ bound with $\beta < 3/2$, there are some with negative energy, which lead to a blow-up in finite time by Glassey-Schaeffer's blow-up theorem (evidently, $\|f_0\|_{3/2} > C_{3/2}$ for those data).*

Remark 1.6. *By the previous two remarks, everything else being equal our sharp $\mathfrak{L}^{3/2}$ condition on f_0 improves on the (nonsharp) $\mathfrak{L}^\infty$ condition on f_0 in [GlSc85].*

Remark 1.7. *If the normalization $\|f_0\|_1 = 1$ is changed to any other value for the $\mathfrak{L}^1$ norm of f_0, the values of the critical C_β for $\beta \geq 3/2$ change by a simple scaling transformation.*

The rest of the paper is structured as follows. In the next section, we list the familiar conservation laws and the virial identities for rVP$^-$. Then, in section 3, we find an $\mathfrak{L}^{3/2}$-optimal subset of $\mathfrak{P}_1 \cap \mathfrak{L}^1(\mathrm{d}p\mathrm{d}q)$ on which the energy functional of f is bounded below, which bound is 0. Section 4 is devoted to obtaining a-priori bounds on the data f_0. In section 5 we prove our global existence and uniqueness result of classical solutions, all of which have strictly positive energy. In section 6, we prove that blow-up in finite time occurs for certain data with non-positive energy. We comment on the critical case in section 7, and section 8 lists some interesting open problems. Finally, in the appendix we give a vindication of rVP$^-$ in terms of certain distributional solutions to two-species neutral rVM.

2 Conservation laws and virial identities

In our paper we will make use of (many of) the conservation laws, of the virial identity, and another identity, all valid a priori for any sufficiently regular solution of rVP$^-$. While (most of) these laws and both identities are proved in [GlSc85] (under more restrictive assumptions than stated here), for the convenience of the reader, these basic results are collected separately in this section. To simplify the notation, we will use the abbreviation $\int$ for $\int_{\mathbb{R}^3}$, and we write

$$\rho(q) := \int f(p,q)\mathrm{d}p \tag{5}$$

for the relative density function in physical space. We shall drop the argument $\mathrm{d}p\mathrm{d}q$ from now on from the symbols for the function spaces.

We begin with the conservation laws for the Casimir functionals of f. Thus, for (the pertinent subset of) $f \in \mathfrak{P} \cap \mathfrak{L}^1$ we define the g-Casimir functional of f by

$$\mathcal{C}^{(g)}(f) = \iint g \circ f \,\mathrm{d}p\mathrm{d}q\,, \quad \text{for all} \quad g : \mathbb{R}_+ \to \mathbb{R} \quad \text{such that} \quad g \circ f \in \mathfrak{L}^1\,. \tag{6}$$

For $g(\,\cdot\,) = (\mathrm{id}\,(\,\cdot\,))^\alpha$, $\alpha \geq 1$, we get the α-th power of the $\mathfrak{L}^\alpha$ norm of f; when $\alpha = 1$ this yields just the mass functional (= integral) of f. The choice $g(\,\cdot\,) = -\mathrm{id}\,(\,\cdot\,)\log(\mathrm{id}\,(\,\cdot\,)/f_*)$ gives the entropy of f relative to some arbitrary $f_* \in \mathfrak{P} \cap \mathfrak{L}^1$,

$$\mathcal{C}^{(-\mathrm{id}\log(\mathrm{id}/f_*))}(f) = -\iint f \ln(f/f_*)\mathrm{d}p\mathrm{d}q \equiv \mathcal{S}(f|f_*)\,. \tag{7}$$

Then, since (1) is isomorphic to a continuity equation for f on $\mathbb{R}^6$, we have

Proposition 2.1. *Let $t \mapsto f_t \in \mathfrak{P} \cap \mathfrak{C}^1$ be a classical solution of rVP$^-$. Then, whenever $\mathcal{C}^{(g)}\big(f_0\big)$ exists, also $\mathcal{C}^{(g)}\left(f_t\right)$ does, and*

$$\mathcal{C}^{(g)}\left(f_t\right) = \mathcal{C}^{(g)}\big(f_0\big)\,. \tag{8}$$

Beside the conservation laws just stated, the familiar quantities energy, momentum, and angular momentum are conserved.

Proposition 2.2. *Let $t \mapsto f_t \in \mathfrak{P}_1 \cap \mathfrak{C}^1$ be a classical solution of rVP$^-$, then the energy of f_t is conserved, i.e. $\mathcal{E}(f_t) = \mathcal{E}(f_0)$, where*

$$\mathcal{E}(f) := \iint \sqrt{1+|p|^2} f(p,q)\mathrm{d}p\,\mathrm{d}q - \frac{1}{2}\iiiint \frac{f(p,q)f(p',q')}{|q-q'|}\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}p'\,\mathrm{d}q'\,, \tag{9}$$

and the momentum space contribution to $\mathcal{E}(f)$ (denoted $\mathcal{E}_p(f)$) is the kinetic plus rest energy, while the physical space contribution (denoted $\mathcal{E}_q(f)$) is the potential energy of f.

Moreover, the momentum of f is conserved, i.e. $\mathcal{P}(f_t) = \mathcal{P}(f_0)$, where

$$\mathcal{P}(f) := \iint p f(p,q)\mathrm{d}p\mathrm{d}q\,, \tag{10}$$

and if also $f_t \in \mathfrak{P}_2$ then so is the angular momentum of f, i.e. $\mathcal{J}(f_t) = \mathcal{J}(f_0)$, where

$$\mathcal{J}(f) := \iint q \times p f(p,q)\mathrm{d}p\mathrm{d}q\,. \tag{11}$$

Beside the angular momentum functional $\mathcal{J}(f)$, also the virial functional

$$\mathcal{V}(f) := \iint q \cdot p f(p,q)\mathrm{d}p\mathrm{d}q \tag{12}$$

plays an important rôle, but it is not conserved. Its time evolution yields what is called the dilation identity for rVP$^-$, which in the physics literature would be part of a “dynamical virial theorem.”

Proposition 2.3. *Let $t \mapsto f_t \in \mathfrak{P}_2 \cap \mathfrak{C}^1$ be a classical solution of rVP$^-$ over the interval $(0,T)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(f_t) = \mathcal{E}(f_t) - \iint \frac{1}{\sqrt{1+|p|^2}} f_t(p,q)\mathrm{d}p\,\mathrm{d}q \tag{13}$$

An immediate and entirely obvious corollary of the dilation identity (13), which nevertheless deserves to be stated in its own right, is the "stationary virial theorem."

Corollary 2.4. *Let $t \mapsto f_t \equiv f_0$ be a stationary solution of rVP$^-$. Then*

$$\mathcal{E}(f_0) = \iint \frac{1}{\sqrt{1+|p|^2}} f_0(p,q)\mathrm{d}p\,\mathrm{d}q\,. \tag{14}$$

Remark 2.5. *If the stationary $f_t = f_0$ has most or all of its mass supported in a cylindrical subset of (p,q) space given by $B_P(0)\times\mathbb{R}^3$ with $P \ll 1$, then we can expand $\sqrt{1+|p|^2} = 1 + \frac{1}{2}|p|^2 + O(|p|^4)$ in (14), both in its r.h.s. and in $\mathcal{E}(f_0)$, and obtain the familiar stationary virial theorem "$2E_{kin} = -E_{pot}$" of non-relativistic VP$^-$, viz.*

$$\iint |p|^2 f_0(p,q)\mathrm{d}p\,\mathrm{d}q = \frac{1}{2}\iint \frac{\rho(q)\rho(q')}{|q-q'|}\mathrm{d}p'\,\mathrm{d}q'\,. \tag{15}$$

Some results will flow entirely from Corollary 2.4. But for the blow-up results we also need another part of the virial theorem, identity (26) in [GlSc85], viz.

Proposition 2.6. *Let $t \mapsto f_t \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ be a classical solution of rVP$^-$ over some time interval $t \in (0,T)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint |q|^2\sqrt{1+|p|^2} f_t(p,q)\mathrm{d}p\,\mathrm{d}q = 2\mathcal{V}(f_t) - \iint |q|^2 v\cdot\nabla_q\phi_t(q) f_t(p,q)\mathrm{d}p\,\mathrm{d}q\,. \tag{16}$$

3 An $\mathfrak{L}^\beta$-optimal f domain for lower boundedness of $\mathcal{E}(f)$

The lower boundedness properties of $\mathcal{E}(f)$ play an important rôle in the proof of our theorem. Clearly, $\mathcal{E}(f)$ is unbounded below on $\mathfrak{P}_1 \cap \mathfrak{L}^1$, for we can make $\mathcal{E}(f)$ as negative as we please along the sequence $f_R \in \mathfrak{P}_1 \cap \mathfrak{L}^1$, given by $f_R(p,q) = (16\pi^2/9)^{-1}R^{-3}\chi_{B_1(0)}(p)\times\chi_{B_R(0)}(q)$, by letting $R \downarrow 0$ (here, χ_S is the characteristic function of the set S). So one needs to restrict $\mathcal{E}(f)$ to some subset of $\mathfrak{P}_1 \cap \mathfrak{L}^1$. Incidentally, though this is not spelled out in the pertinent references, it follows from inequalities (5) and (22) in [GlSc85] that $\mathcal{E}(f)$ is bounded below (by 0, then) when $f \in \mathfrak{P}_1 \cap \mathfrak{L}^\infty$ with $\|f\|_\infty$ sufficiently small, and it follows from the displayed but unnumbered inequality in the introduction of [HaRe07] that $\mathcal{E}(f) \geq 0$ also when $f \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta$ with $\beta \geq 3/2$ and $\|f\|_\beta$ small enough. Neither of these references reveal how small is "small enough" and what happens for larger norms or smaller β.

We here are interested in the $\mathfrak{L}^\beta$-optimal domain in f space, in the sense that for $f \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta$ we seek the smallest possible β, and the largest possible $\mathfrak{L}^\beta$ norm of f, such that lower boundedness of $\mathcal{E}(f)$ holds, while $\mathcal{E}(f)$ is unbounded below when these conditions on f are not met. If the Laplacian in Poisson's equation (3) for ϕ is replaced by the d'Alembertian, then $\beta = 3/2$ is the optimal β value for the corresponding "relativistic Vlasov-d'Alembert" (rVdA) energy functional, and the sharp value of the critical $\mathfrak{L}^{3/2}$ norm of f can by computed.[4] Curiously, the energy functional $\mathcal{E}(f)$ of rVP$^-$ is bounded below iff the rVdA energy functional is, and by the same bound then, even though $\mathcal{E}(f)$ could have been better behaved, a priori speaking, because in the rVdA energy functional f and ϕ represent independent degrees of freedom. To exhibit these aspects very clearly, instead of elaborating on the approach of [HaRe07], which is based on the Hardy–Littlewood–Sobolev inequality and standard interpolation arguments and which would also yield the sharp $\mathfrak{L}^{3/2}$ bound, we investigate the boundedness of $\mathcal{E}(f)$ in terms of the topologically dual approach which can be applied almost verbatim also when the Poisson equation is replaced by the inhomogeneous wave equation.

Thus we introduce the following functional of f and ϕ,

$$\widetilde{\mathcal{E}}(f,\phi) := \iint \left(\sqrt{1+|p|^2} + \phi(q)\right) f(p,q)\mathrm{d}p\,\mathrm{d}q + \frac{1}{8\pi}\int \left|\nabla_q \phi\right|^2(q)\mathrm{d}q\,, \tag{17}$$

irrespective of whether the functions f and ϕ are related by the Poisson equation (3) or not, and of whether ϕ satisfies (4) or not. If f and ϕ are related by $\phi = -|\mathrm{id}\,|^{-1} * \int f \mathrm{d}p$, as for solutions to rVP$^-$, then (9) and (17) can be converted into one another by using the Poisson equation (3) and an integration by parts; in those cases $\mathcal{E}(f) = \min_{\psi\in\dot{\mathfrak{H}}^1_0} \widetilde{\mathcal{E}}(f,\psi)$, and ϕ is the minimizer.

Proposition 3.1. *Let* $\Omega = \{(f,\phi)\big| f \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2},\ \phi \in \dot{\mathfrak{H}}^1_0\}$. *Then*

$$\inf\left\{\widetilde{\mathcal{E}}(f,\phi)\Big|(f,\phi)\in\Omega, \|f\|_{3/2} \leq (3/8)(15/16)^{1/3}\right\} = 0. \tag{18}$$

The $\mathfrak{L}^{3/2}$ *bound on* f *is sharp in the sense that, for any* $\epsilon > 0$, $C > 0$, *we can find* $f \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$ *with* $\|f\|_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}(1+\epsilon)$, *and* $\phi \in \dot{\mathfrak{H}}^1_0$ *such that* $\widetilde{\mathcal{E}} \leq -C$. *In particular,* $\widetilde{\mathcal{E}}$ *is unbounded below if the* $\mathfrak{L}^{3/2}$ *bound on* f *is replaced by any* $\mathfrak{L}^\beta$ *bound for any* $\beta < 3/2$. *Moreover, the infimum (18) is not a minimum.*

[4] Unpublished math. phys. seminar talk at the E. Schrödinger Inst., Vienna, Aug. 02, 2002. M.K. takes the opportunity to thank N. Mauser for his kind invitation to present these results.

Proof of Proposition 3.1. Using the estimates $|p| < \sqrt{1+|p|^2} \leq 1+|p|$, we have $\widetilde{\mathcal{K}} < \widetilde{\mathcal{E}} \leq 1 + \widetilde{\mathcal{K}}$, where

$$\widetilde{\mathcal{K}}(f,\phi) := \frac{1}{8\pi}\int \left|\nabla_q \phi\right|^2(q)\mathrm{d}q + \iint \Big(|p| + \phi(q)\Big) f(p,q)\mathrm{d}p\,\mathrm{d}q\,, \tag{19}$$

so that to prove the lower boundedness vs. unboundedness of $\widetilde{\mathcal{E}}(f,\phi)$ as claimed, it basically suffices to work with $\widetilde{\mathcal{K}}(f,\phi)$. Only for the precise value of the infimum do we need one extra estimate involving $\sqrt{1+|p|^2}$. We also introduce the abbreviation $h(p,q) := |p| + \phi(q)$ for this auxiliary, "ultra-relativistic" single-particle Hamiltonian, and we define $\mathcal{E}^u_p(f) := \iint |p| f(p,q)\mathrm{d}p\mathrm{d}q$.

To prove boundedness below of $\widetilde{\mathcal{K}}(f,\phi)$ (hence, of $\widetilde{\mathcal{E}}(f,\phi)$) on the subset of Ω for which $\|f\|_{3/2} \leq \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$, we begin by noting that unboundedness below of $\widetilde{\mathcal{K}}$ can only occur if $\phi(q) < 0$ for some q; hence, we only need to show that the lesser functional $\underline{\widetilde{\mathcal{K}}} \leq \widetilde{\mathcal{K}}$ given by

$$\underline{\widetilde{\mathcal{K}}}(f,\phi) = \frac{1}{8\pi}\int \left|\nabla_q \phi\right|^2(q)\mathrm{d}q - \iint h_-(p,q) f(p,q)\mathrm{d}p\,\mathrm{d}q \tag{20}$$

is bounded below, where $h_- := -\min\{h, 0\} \geq 0$ is the negative part of h. To this effect we now apply Hölder's inequality to $\int h_- f \mathrm{d}p\mathrm{d}q$, obtaining the estimate

$$\underline{\widetilde{\mathcal{K}}}(f,\phi) \geq \frac{1}{8\pi}\int \left|\nabla_q \phi\right|^2(q)\mathrm{d}q - \|h_-\|_\tau \|f\|_{\tau/(\tau-1)}\,, \tag{21}$$

with τ still to be determined. A simple integration with spherical coordinates in p space gives

$$\|h_-\|_\tau^\tau = \frac{8\pi}{\prod_{k=1}^3 (k+\tau)} \|\phi_-\|_{3+\tau}^{3+\tau} \tag{22}$$

whenever $\|\phi_-\|_{3+\tau}$ exists. Now the Sobolev embedding says[5] $\dot{\mathfrak{H}}^1_0(\mathbb{R}^3) \to \mathfrak{L}^\alpha(\mathbb{R}^3)$ iff $\alpha = 6$. By (22), this means $\tau = 3$, yielding $\tau/(\tau-1) = 3/2$ and $\|h_-\|_3^3 = (\pi/15)\|\phi_-\|_6^6$. Using next the inclusion[6] $\operatorname{supp}\phi_- \subseteq \operatorname{supp}\phi$, we find the estimate

$$\|h_-\|_3 \leq \left(\frac{\pi}{15}\right)^{1/3} \|\phi\|_6^2\,. \tag{23}$$

[5]Note that we work with the homogeneous norm; the standard $\mathfrak{H}^1$ embedding of course gives $\mathfrak{H}^1(\mathbb{R}^3) \to \mathfrak{L}^\alpha(\mathbb{R}^3)$ for all $\alpha \in [2,6]$.

[6]Note that these fine details are necessary only because we did not restrict ϕ by Poisson's equation (3) with asymptotic condition (4); for if we had, then $\phi = -\phi_-$ would ensue.

Hence, we set $\tau = 3$ in (21) and use (23), next recall the sharp Sobolev inequality for the embedding $\dot{\mathfrak{H}}^1_0 \to \mathfrak{L}^6$ (see [Tal76, Lie83]), viz.

$$\left\|\nabla_q \phi\right\|_2^2 - 3\left(\frac{\pi}{2}\right)^{4/3} \|\phi\|_6^2 \geq 0\,, \tag{24}$$

where $3(\pi/2)^{4/3}$ is the largest possible coefficient for the $\|\phi\|_6^2$ term, and obtain

$$\widetilde{\mathcal{K}}(f,\phi) \geq \left(\frac{3}{8}\left(\frac{\pi}{16}\right)^{1/3} - \left(\frac{\pi}{15}\right)^{1/3} \|f\|_{3/2}\right) \|\phi\|_6^2\,. \tag{25}$$

Thus we have proved that

$$\inf\left\{\widetilde{\mathcal{E}}(f,\phi)\middle|(f,\phi)\in\Omega, \|f\|_{3/2} \leq (3/8)(15/16)^{1/3}\right\} \geq 0. \tag{26}$$

To see that 0 is the best lower bound for $\widetilde{\mathcal{E}}$ on Ω when $\|f\|_{3/2} \leq \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$, we work with trial densities of the type $f^\phi(p,q) := h_-^2(p,q)/\|h_-\|_2^2$, with nonpositive $\phi \in \dot{\mathfrak{H}}^1_0$ still to be chosen. Clearly, $f^\phi \geq 0$ and $\left\|f^\phi\right\|_1 = 1$, and furthermore$\widetilde{\mathcal{K}}(f^\phi,\phi) = \underline{\widetilde{\mathcal{K}}}(f^\phi,\phi)$. We easily compute that

$$\iint h(p,q) f^\phi(p,q)\mathrm{d}p\mathrm{d}q = -\frac{\|h_-\|_3^3}{\|h_-\|_2^2} \tag{27}$$

and $\left\|f^\phi\right\|_{3/2} = \|h_-\|_3^2/\|h_-\|_2^2$. With the help of (22) and $\phi \leq 0$, we now find $\|h_-\|_3^3 = (\pi/15)\|\phi\|_6^6$, and we readily calculate that $\|h_-\|_2^2 = (2\pi/15)\|\phi\|_5^5$, so that the trial $\phi(q)$ will have to approach 0 sufficiently fast as $|q| \to \infty$ in order for $\|\phi\|_5$ to exist; in particular, (4) is fast enough. For such f^ϕ, we therefore have

$$\iint h(p,q) f^\phi(p,q)\mathrm{d}p\mathrm{d}q = -\frac{1}{2}\frac{\|\phi\|_6^6}{\|\phi\|_5^5} \tag{28}$$

and

$$\left\|f^\phi\right\|_{3/2} = \frac{1}{2}\left(\frac{15}{\pi}\right)^{1/3} \frac{\|\phi\|_6^4}{\|\phi\|_5^5}\,, \tag{29}$$

so that

$$\iint h(p,q) f^\phi(p,q)\mathrm{d}p\mathrm{d}q = -\left(\frac{\pi}{15}\right)^{1/3} \|f\|_{3/2}\|\phi\|_6^2. \tag{30}$$

By taking strong limits we now see that (30) is valid for all $f \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$ and $\phi \in \mathfrak{L}^6$. In total we have $\widetilde{\mathcal{K}}(f^\phi, \phi) = \mathcal{G}(\phi)$, with

$$\mathcal{G}(\phi) := \frac{1}{8\pi}\big\|\nabla_q \phi\big\|_2^2 - \Big(\frac{\pi}{15}\Big)^{1/3}\big\|f^\phi\big\|_{3/2}\|\phi\|_6^2 . \tag{31}$$

Now recall that up to translations, the scaling family of functions

$$\phi_\kappa(q) = -\frac{\kappa}{\sqrt{1+\kappa^2|q|^2}} \tag{32}$$

with $\kappa > 0$ provides us with all the optimizers satisfying (4) of the sharp Sobolev inequality (24) for the embedding $\dot{\mathfrak{H}}_0^1 \to \mathfrak{L}^6$, i.e. "=" holds in (24) when $\phi = \phi_\kappa$; see [Tal76, Lie83]. We set $\phi = \phi_\kappa$ in f^ϕ; note that $f^{\phi_\kappa} \in \mathfrak{P}_1 \cap \mathfrak{C}^\infty$, so $f^{\phi_\kappa} \in \mathfrak{P}_1 \cap \mathfrak{L}^\alpha$ for all α. An easy computation gives us $\|\phi_\kappa\|_6^4/\|\phi_\kappa\|_5^5 = 3\pi^{1/3}/4^{5/3}$, independent of κ. Hence, by (29), $\big\|f^{\phi_\kappa}\big\|_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$ independent of κ. Therefore $\mathcal{G}(\phi_\kappa) = 0$ for all κ, and since $\sup_{|p|\geq 0}\big(|p|\sqrt{1+|p|^2} - |p|^2\big) = 1/2$, we now conclude that for any small $\epsilon > 0$ we have

$$\begin{aligned}
\widetilde{\mathcal{E}}(f^{\phi_\kappa}, \phi_\kappa) &= \iint \Big(\sqrt{1+|p|^2} - |p|\Big) f^{\phi_\kappa}(p,q)\mathrm{d}p\,\mathrm{d}q \\
&\leq \frac{1}{2}\iint |p|^{-1} f^{\phi_\kappa}(p,q)\mathrm{d}p\,\mathrm{d}q \\
&= \frac{5}{4}\frac{\|\phi_\kappa\|_4^4}{\|\phi_\kappa\|_5^5} = C_*\frac{1}{\kappa} < \epsilon
\end{aligned} \tag{33}$$

whenever $\kappa > C_*/\epsilon$, where C_* is independent of κ. The "inf-part" (18) of our Proposition 3.1 is proved.

Incidentally, with a minor modification, the last line of reasoning also shows that "inf $\neq$ min." Thus, for any $(f,\phi) \in \Omega$ satisfying $\|f\|_{3/2} \leq \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$, we have

$$\begin{aligned}
\widetilde{\mathcal{E}}(f,\phi) &= \iint \Big(\sqrt{1+|p|^2} - |p|\Big) f(p,q)\mathrm{d}p\,\mathrm{d}q + \widetilde{\mathcal{K}}(f,\phi) \\
&\geq \iint \Big(\sqrt{1+|p|^2} - |p|\Big) f(p,q)\mathrm{d}p\,\mathrm{d}q \ > 0 .
\end{aligned} \tag{34}$$

Next, to prove that the $\mathfrak{L}^{3/2}$ bound on f is sharp, we now show that for any given $\epsilon > 0$ and $C > 0$, we can find $f \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$ and $\phi \in \dot{\mathfrak{H}}_0^1$ such that

$\|f\|_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}(1+\epsilon)$ and $\widetilde{\mathcal{E}} \leq -C$. As stated at the beginning of the proof of our proposition, it suffices to prove unboundedness below for $\widetilde{\mathcal{K}}$ along these lines. For this purpose we continue to work with the optimizers of Sobolev's inequality (24), ϕ_κ, but now invoke the double scaling family of trial densities $f = f^{\phi_1}_{\kappa,\lambda}$, $\kappa > 0, \lambda > 0$, defined by

$$f^{\phi_1}_{\kappa,\lambda}(p,q) := \kappa^3\lambda^3 f^{\phi_1}(\lambda p, \kappa q)\,. \tag{35}$$

Note that $f^{\phi_1}_{\kappa,\lambda} \in \mathfrak{P}_1 \cap \mathfrak{C}^\infty$ for all κ, λ, so $f^{\phi_1}_{\kappa,\lambda} \in \mathfrak{P}_1 \cap \mathfrak{L}^\alpha$ for all α. Note furthermore by an obvious re-scaling of the integration variable that the pertinent density in q space is independent of λ, i.e. $\int f^{\phi_1}_{\kappa,\lambda}(p,q)\mathrm{d}p = \int f^{\phi_1}_{\kappa,1}(p,q)\mathrm{d}p$ for all λ; moreover, it equals $\int f^{\phi_\kappa}(p,q)\mathrm{d}p =: \rho^{\phi_\kappa}(q)$, for $f^{\phi_1}_{\kappa,1/\kappa} = f^{\phi_\kappa}$. The density $\rho^{\phi_\kappa}(q) = \kappa^3\rho^{\phi_1}(\kappa q)$, where ρ^{ϕ_1} is given by

$$\rho^{\phi_1}(q) = \frac{3}{4\pi}(-\phi_1)^5(q)\,. \tag{36}$$

Thus, not only are $f^{\phi_\kappa}(p,q)$ and $\phi_\kappa(q)$ related by Poisson's equation (3) with (4), but so are $f^{\phi_1}_{\kappa,\lambda}(p,q)$ and $\phi_\kappa(q)$ for all λ. Next, a simple computation yields

$$\big\|f^{\phi_1}_{\kappa,\lambda}\big\|_{3/2} = \kappa\lambda\big\|f^{\phi_1}\big\|_{3/2}\,, \tag{37}$$

with $\big\|f^{\phi_1}\big\|_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$, so setting $\big\|f^{\phi_1}_{\kappa,\lambda}\big\|_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}(1+\epsilon)$ in (37) defines a branch of a hyperbola $\kappa\lambda = 1+\epsilon$ in the first quadrant of Cartesian (κ,λ)-parameter space. Along this hyperbola branch, $f^{\phi_1}_{\kappa,\lambda}$ is a probability density with $\mathfrak{L}^{3/2}$ norm equal to $(1+\epsilon)C_{3/2}$, i.e. bigger than the acclaimed critical $C_{3/2}$. Next, a straightforward computation shows that along this hyperbola branch,

$$\widetilde{\mathcal{K}}\big(f^{\phi_1}_{\kappa,(1+\epsilon)/\kappa}, \phi_\kappa\big) = -\frac{3\pi}{32}\frac{\epsilon}{1+\epsilon}\kappa < 0 \tag{38}$$

for all $\kappa > 0$. Hence, for any $\epsilon > 0$ and $C > 0$ there is a unique $\kappa(\epsilon, C) = (32/3\pi)(1+1/\epsilon)C > 0$, such that $\widetilde{\mathcal{K}}\big(f^{\phi_1}_{\kappa,(1+\epsilon)/\kappa}, \phi_\kappa\big) < -C$ whenever $\kappa > \kappa(\epsilon, C)$. Therefore, $\widetilde{\mathcal{K}}\big(f^{\phi_1}_{\kappa,(1+\epsilon)/\kappa}, \phi_\kappa\big)$ is unbounded below, and this implies unboundedness below of $\widetilde{\mathcal{K}}(f,\phi)$ over the set $\{(f,\phi) \in \Omega,\ \|f\|_{3/2} \leq C\}$ when $C > \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$; hence, the same holds for $\widetilde{\mathcal{E}}(f,\phi)$ $[\leq 1 + \widetilde{\mathcal{K}}(f,\phi)]$.

Finally we prove that $\widetilde{\mathcal{K}}(f,\phi)$ is unbounded below on the domain $\{\phi \in \dot{\mathfrak{H}}^1_0, f \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta, \|f\|_\beta \leq C_\beta\}$ for any $\beta \in (1,3/2)$ and any $C_\beta > 0$. (Notice that for $\beta = 1$, which only allows $\|f\|_1 = 1$, we already proved unboundedness at the beginning of section 3.) We ignore (3), as allowed by the hypotheses of Proposition 3.1. Thus, since for $\beta < 3/2$ the interval $(2, 3/\beta)$ is not empty, we pick any $\vartheta \in (2, 3/\beta)$. We now choose a family of trial densities of the type $\hat{f}^\phi(p,q) := h_-^\vartheta(p,q)/\|h_-\|_\vartheta^\vartheta$, with some nonpositive $\phi \in \dot{\mathfrak{H}}^1_0$ satisfying (4); ϕ will be specified further below. Also for this trial family of fs we have $\widetilde{\mathcal{K}}(\hat{f}^\phi,\phi) = \underline{\widetilde{\mathcal{K}}}(\hat{f}^\phi,\phi)$. Moreover, using again (22), we find

$$\left\|\hat{f}^\phi\right\|_\beta = a(\vartheta,\beta)\frac{\|\phi\|_{\vartheta\beta+3}^{\vartheta+3/\beta}}{\|\phi\|_{\vartheta+3}^{\vartheta+3}}\,, \tag{39}$$

and

$$\widetilde{\mathcal{K}}(\hat{f}^\phi,\phi) = \frac{1}{8\pi}\left\|\nabla_q\phi\right\|_2^2 - b(\vartheta)\frac{\|\phi\|_{\vartheta+4}^{\vartheta+4}}{\|\phi\|_{\vartheta+3}^{\vartheta+3}}\,, \tag{40}$$

where a and b are some numerical constants dependent on the displayed arguments. Now notice that for the stipulated range of β and ϑ values, we have $5 < \vartheta + 3 < 6$ and $5 < \vartheta\beta + 3 < 6$, but $\vartheta + 4 > 6$. Since for $\phi \in \dot{\mathfrak{H}}^1_0$ satisfying (4) we actually have $\phi \in \mathfrak{L}^\alpha(\mathbb{R}^3)$ for all $\alpha \in (3,6]$, it follows right away that $\left\|\hat{f}^\phi\right\|_\beta < \infty$. But among these ϕ functions there are infinitely many for which $\|\phi\|_{\vartheta+4}^{\vartheta+4} = \infty$. Let ϕ_* be such a ϕ. We then can make $\left\|\hat{f}^{\phi_*}\right\|_\beta \leq C_\beta$ for any small C_β by multiplying ϕ_* by some small constant c if necessary (note that $c\phi_*$ no longer satisfies (4), though), while $\|\phi_*\|_{\vartheta+4}^{\vartheta+4} = \infty$ implies $\widetilde{\mathcal{K}}(\hat{f}^{\phi_*},\phi_*) = -\infty$. Therefore, we obtain lower unboundedness of $\widetilde{\mathcal{E}}(f,\phi)$ on the domain $\{\phi \in \dot{\mathfrak{H}}^1_0, f \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta, \|f\|_\beta \leq C_\beta\}$ for any $1 < \beta < 3/2$ and any C_β. So $\mathfrak{L}^{3/2}$ in (18) cannot be replaced by any $\mathfrak{L}^\beta$ with $\beta < 3/2$. Q.E.D.

Remark 3.2. *Our proof, a variant of which was announced a while ago (see footnote 4), was inspired by a related proof in [Aly89] (which in turn was inspired by one in [WZS88]) for the energy functional of the nonrelativistic VP system with $\sigma = -1$. For VP⁻ the critical $\beta = 9/7$, and the critical $\mathfrak{L}^{9/7}$ bound on f simply reads $\|f\|_{9/7} < \infty$.*

We are now ready to address the boundedness of $\mathcal{E}(f)$.

Proposition 3.3. *We have*

$$\inf\left\{\mathcal{E}(f)\middle|f\in\mathfrak{P}_1\cap\mathfrak{L}^{3/2},\ \|f\|_{3/2}\le(3/8)(15/16)^{1/3}\right\}=0\,, \tag{41}$$

and the infimum (41) is not a minimum. Moreover, we have

$$\inf\left\{\mathcal{E}(f)\middle|f\in\mathfrak{P}_1\cap\mathfrak{L}^{\beta},\ \|f\|_{\beta}\le C\right\}=-\infty \tag{42}$$

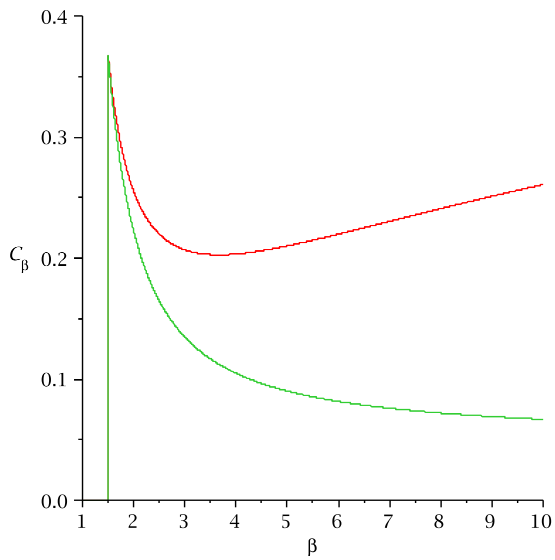
whenever $C>C_\beta$, with

$$C_\beta:=\inf_{\mathfrak{P}_1\cap\mathfrak{L}^\beta}\left(\frac{\mathcal{E}^u_p(f)}{-\mathcal{E}_q(f)}\right)^{3(1-1/\beta)}\|f\|_\beta\,. \tag{43}$$

In particular, $C_\beta=0$ for all $\beta\in(1,3/2)$. Furthermore, for all $\beta\ge 3/2$ we have

$$\left[\left(\frac{3}{8}\right)^3\frac{15}{16}\right]^{1-1/\beta}\le C_\beta\le\frac{45}{8\pi^2}\left(\frac{8\pi^{5/2}}{\prod_{k=1}^3(k+2\beta)}\frac{\Gamma(\beta)}{\Gamma(\beta+\frac{3}{2})}\right)^{1/\beta}. \tag{44}$$

At $\beta=3/2$ the upper and lower bounds in (44) coincide; $C_{3/2}=(3/8)(15/16)^{1/3}$ is the sharp value, i.e. the bound on the $\mathfrak{L}^{3/2}$ norm of f in (41) is optimal.



Upper and lower bounds on C_β together with the vertical line segment between 0 and $C_{3/2}$. Replacing the co-ordinate label C_β by $\|f\|_\beta$, the diagram acquires the following meaning: On and to the right of the vertical line segment, yet on and below the lower curve, we have $\inf\mathcal{E}(f)\ge 0$, while $\inf\mathcal{E}(f)=-\infty$ to the left of the vertical line and above the upper curve.

Proof of Proposition 3.3. As to (41), since the Poisson equation (3) together with (4) selects a subset of Ω, it follows that $\mathcal{E}(f)$ is bounded below whenever $\widetilde{\mathcal{E}}(f,\phi)$ is. Furthermore, since we proved the infimum versus unboundedness of $\widetilde{\mathcal{E}}(f,\phi)$ when $\phi \in \dot{\mathfrak{H}}^1_0$ and $f \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$ with pairs of trial functions $(f^{\phi_\kappa}, \phi_\kappa)$ respectively $(f^{\phi_1}_{\kappa,\lambda}, \phi_\kappa)$ (with ϕ_κ given by (32)), either pair of which solves Poisson's equation (3) together with (4), the same infimum versus unboundedness features hold for $\mathcal{E}(f)$ when $f \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$; viz. (41) is optimal, indeed, and "inf $\neq$ min."

As to (42) and (43), we set $\mathcal{K}(f) := \mathcal{E}^u_p(f) + \mathcal{E}_q(f)$. As with $\widetilde{\mathcal{K}}(f,\phi)$ and $\widetilde{\mathcal{E}}(f,\phi)$, unboundedness below of $\mathcal{K}(f)$ implies unboundedness below of $\mathcal{E}(f)$. In this vein, by double scaling we find $\mathcal{K}(f_{\kappa,\lambda}) = \lambda^{-1}\mathcal{E}^u_p(f) + \kappa\mathcal{E}_q(f)$, so for any f we can give $\mathcal{K}(f_{\kappa,\lambda})$ any value we like by choosing κ and λ appropriately, at the expense of changing $\|f\|_\beta$ to $\|f_{\kappa,\lambda}\|_\beta = (\kappa\lambda)^{3(1-1/\beta)}\|f\|_\beta$. In particular, once we found a special (κ_0,λ_0) pair for which $\mathcal{K}(f_{\kappa_0,\lambda_0}) < 0$, we can then let $\mathcal{K}(f_{\kappa,\lambda}) \downarrow -\infty$ by scaling along the branch of a hyperbola $\kappa\lambda = \kappa_0\lambda_0$ in the first quadrant of Cartesian (κ,λ)-parameter space, keeping $\|f_{\kappa,\lambda}\|_\beta$ fixed along this scaling sequence. For each f, the borderline case $\mathcal{K}(f_{\kappa,\lambda}) = 0$ is obtained by choosing $\kappa\lambda = -\mathcal{E}^u_p(f)/\mathcal{E}_q(f)$. Hence, (42) holds whenever $C > C_\beta$, with C_β the infimum over $f \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta$ of $\|f_{\kappa,\lambda}\|_\beta = (-\mathcal{E}^u_p(f)/\mathcal{E}_q(f))^{3(1-1/\beta)}\|f\|_\beta$, i.e. (43).

As for (44), since $(f^{\phi_1}_{\kappa,\lambda}, \phi_\kappa)$ jointly solves Poisson's equation (3) with (4), our proof of unboundedness of $\widetilde{\mathcal{E}}(f,\phi)$ when $\left\|f^{\phi_1}_{\kappa,\lambda}\right\|_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}(1+\epsilon)$ applies nearly verbatim, with $\left\|f^{\phi_1}_{\kappa,\lambda}\right\|_{3/2}$ now replaced by $\left\|f^{\phi_1}_{\kappa,\lambda}\right\|_\beta$ and $\frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$ $(= \left\|f^{\phi_1}\right\|_{3/2} = C_{3/2})$ replaced by $\left\|f^{\phi_1}\right\|_\beta =$ r.h.s.(44), giving (42) whenever $C >$ r.h.s.(44), implying the second inequality in (44). The first inequality in (44) follows from the interpolation inequality and (41), cf. our Remark 1.4.

Lastly, to prove $\inf \mathcal{E}(f) = -\infty$ when $\beta \in (1, 3/2)$, we use $\mathcal{E}(f) = \min_{\phi\in\dot{\mathfrak{H}}^1_0} \widetilde{\mathcal{E}}(f,\phi)$ and the fact that $\inf_\Omega \widetilde{\mathcal{E}}(f,\phi) = -\infty$ for this β interval (for which proof we conveniently ignored (3) and (4)). Yet, it is desirable to also have a direct proof for $\mathcal{E}(f)$.

We need to distinguish between $\beta \in (1, 6/5)$ and $\beta \in [6/5, 3/2)$. First, when $\beta < 6/5$, then by taking product densities $f(p,q) = g(p)\rho(q)$ we have $\rho \in \mathfrak{L}^\beta$ for $\beta < 6/5$, and then it is well-known that we can find f with $\mathcal{E}^u_p(f) < \infty$ and $\|f\|_\beta < \infty$ but with $\mathcal{E}_q(f) = -\infty$. Hence $C_\beta = 0$ when $\beta \in (1, 6/5)$.

Next, when $\beta \in [6/5, 3/2)$, we can recycle the trial densities of the type $\hat{f}^\phi(p,q) = h^\vartheta_-(p,q)/\|h_-\|^\vartheta_\vartheta$, introduced in the proof of Proposition 3.1, but now with $\phi(q)$ re-

placed by $\psi_\delta(q) = -e^{-|q|}/|q|^\delta$, where $\delta > 0$ will be specified below; also $\vartheta > 0$ will need to be re-specified as well. In particular, ψ_δ here is *not* the self-consistent potential ϕ of rVP$^-$, which ϕ is related to $\hat{f}$ by Poisson's equation. Rather, ψ_δ means a convenient way of generating a suitable family of densities $\rho = \hat{\rho}^{\psi_\delta} = (-\psi_\delta)^{\vartheta+3}/\|\psi_\delta\|_{\vartheta+3}^{\vartheta+3}$ which exhibit a local power law singularity; the exponential factor serves to avoid integrability problems at spatial infinity that could occur with these power laws. We will show that for $\beta \in [6/5, 3/2)$ we can find $\delta > 0$ and $\vartheta > 0$ such that $\hat{f}^{\psi_\delta} \in \mathfrak{L}^\beta$ but $\hat{f}^{\psi_\delta} \notin \mathfrak{L}^{3/2}$, and such that $\mathcal{E}_p^u(\hat{f}^{\psi_\delta}) < \infty$ while $\mathcal{E}_q(f) = -\infty$.

Indeed, since $\left\|\hat{f}^{\psi_\delta}\right\|_\beta$ is given by (39) with ϕ replaced by ψ_δ, we have $\hat{f}^{\psi_\delta} \notin \mathfrak{L}^{3/2}$ yet $\hat{f}^{\psi_\delta} \in \mathfrak{L}^\beta$ for any given $\beta \in [6/5, 3/2)$ whenever $\delta \in [2/(2+\vartheta), 3/(3+\vartheta\beta))$, which interval is not empty when $\vartheta > 0$. For the just established range of δ values, given any $\beta \in [6/5, 3/2)$ and for any $\vartheta > 0$, we also have $\hat{\rho}^{\psi_\delta} \in \mathfrak{P}_1 \cap \mathfrak{L}^1$. (All of the above holds also with $\beta \in (1, 3/2)$.) Next, since $\rho \in \mathfrak{L}^{6/5}$ implies $\mathcal{E}_q(f) > -\infty$, we certainly want $\hat{\rho}^{\psi_\delta} \notin \mathfrak{L}^{6/5}$, which means we want $\left\|\hat{\rho}^{\psi_\delta}\right\|_{6/5} = \|\psi_\delta\|_{6(\vartheta+3)/5}^{(\vartheta+3)}/\|\psi_\delta\|_{\vartheta+3}^{\vartheta+3}$ to be ∞. This gives us the condition $\delta \in [5/(6+2\vartheta), 3/(3+\vartheta))$, which interval is not empty for any $\vartheta > 0$. Taking the intersection of our two δ intervals we find the common condition $\delta \in [\max\{2/(2+\vartheta), 5/(6+2\vartheta)\}, 3/(3+\vartheta\beta))$, which interval is not empty for $\beta \in [6/5, 3/2)$ when $\vartheta \in (0, 3/(5\beta-6))$; note that $5/(6+2\vartheta) < 2/(2+\vartheta)$ for $\vartheta \in (0, 2)$ and $5/(6+2\vartheta) > 2/(2+\vartheta)$ for $\vartheta > 2$. Lastly, we want that $\mathcal{E}_p^u(\hat{f}^{\psi_\delta}) < \infty$. Since $\mathcal{E}_p^u(\hat{f}^{\psi_\delta}) = b(\vartheta)\|\psi_\delta\|_{\vartheta+4}^{\vartheta+4}/\|\psi_\delta\|_{\vartheta+3}^{\vartheta+3}$, the necessary and sufficient condition for $\mathcal{E}_p^u(\hat{f}^{\psi_\delta}) < \infty$ is $\delta < 3/(4+\vartheta)$, which may imply, or be implied by, $\delta < 3/(3+\vartheta\beta)$, depending on β; moreover, $3/(4+\vartheta) > \max\{2/(2+\vartheta), 5/(6+2\vartheta)\}$ iff $\vartheta > 2$, but in that case $\max\{2/(2+\vartheta), 5/(6+2\vartheta)\} = 5/(6+2\vartheta)$. Hence, the resulting common intersection of all our δ intervals is the interval $[5/(6+2\vartheta), \min\{3/(3+\vartheta\beta), 3/(4+\vartheta)\})$, which is not empty for $\beta \in [6/5, 3/2)$ when $\vartheta \in (2, 3/(5\beta-6))$. In summary, for $\beta \in [6/5, 3/2)$, when $\vartheta \in (2, 3/(5\beta-6))$ and $\delta \in [5/(6+2\vartheta), \min\{3/(3+\vartheta\beta), 3/(4+\vartheta)\})$, then we have that $\hat{f}^{\psi_\delta} \in \mathfrak{L}^\beta$ with $\beta \in [6/5, 3/2)$ but $\hat{f}^{\psi_\delta} \notin \mathfrak{L}^{3/2}$; we also have $\mathcal{E}_p^u(\hat{f}^{\psi_\delta}) < \infty$; furthermore, the pertinent $\hat{\rho}^{\psi_\delta} \notin \mathfrak{L}^{6/5}$. A brief calculation shows for $\hat{\phi}(|q|) = -|\mathrm{id}|^{-1} * \int \hat{f}^{\psi_\delta}(p,q)\mathrm{d}p$ that $\hat{\phi}'(|q|) \asymp |q|^{1-\delta(3+\vartheta)}$ near the singularity, so $\hat{\phi} \notin \dot{H}^1$ when $\delta \geq 3/(6+2\vartheta)$. Thus, $\delta \geq 5/(6+2\vartheta)$ implies $\mathcal{E}_q(\hat{f}^{\psi_\delta}) = -\infty$, and so

$$C_\beta = \left(\frac{\mathcal{E}_p^u(\hat{f}^{\psi_\delta})}{-\mathcal{E}_q(\hat{f}^{\psi_\delta})}\right)^{3(1-1/\beta)} \left\|\hat{f}^{\psi_\delta}\right\|_\beta = 0 \quad \text{when} \quad \beta \in [6/5, 3/2). \tag{45}$$

Indeed, $\mathcal{K}(\hat{f}^{\psi_\delta}) = -\infty$, which implies $\mathcal{E}(\hat{f}^{\psi_\delta}) = -\infty$. Q.E.D.

Definition 3.4. *In the following, we will call initial data $f_0 \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$* ***subcritical*** *if $\|f_0\|_{3/2} < C_{3/2}$,* ***critical*** *if $\|f_0\|_{3/2} = C_{3/2}$, and* ***supercritical*** *if $\|f_0\|_{3/2} > C_{3/2}$. We use the analogous terminology for the solutions launched by such data.*

4 A-priori bounds for subcritical f

By (34), for all critical and subcritical f we have that $\mathcal{E}(f) = \widetilde{\mathcal{E}}(f, \phi[f]) > 0$, where

$$\phi[f] = -|\mathrm{id}\,|^{-1} * \int f \mathrm{d}p\,. \tag{46}$$

Unfortunately, this does not seem to lend itself to further estimates on the individual energy contributions $\mathcal{E}_p(f)$ and $\mathcal{E}_q(f)$. Yet, for subcritical f we actually have a better result than (34), and this does lead to a-priori bounds on $\mathcal{E}_p(f)$ and $\mathcal{E}_q(f)$.

Proposition 4.1. *For any $f \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$ satisfying $\|f\|_{3/2} < C_{3/2}$, we have*

$$\mathcal{E}(f) \geq \frac{1}{8\pi\varkappa(f)} \int \left|\nabla_q \phi[f]\right|^2(q)\mathrm{d}q \tag{47}$$

with

$$\varkappa(f) = \frac{C_{3/2}}{C_{3/2} - \|f\|_{3/2}}\,. \tag{48}$$

Proof of Proposition 4.1. Since by hypothesis $\|f\|_{3/2}$ is strictly less than the sharp critical value, we can retain a little bit from the Dirichlet integral and see, by inspecting the steps of the proof of the lower boundedness of $\widetilde{\mathcal{E}}$, hence of $\mathcal{E}$, that now we get the estimate (47). Q.E.D.

Proposition 4.1 has the following important spin-off.

Corollary 4.2. *Let $f \in \mathfrak{P}_1 \cap \mathfrak{L}^{3/2}$ satisfy $\|f\|_{3/2} < \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$. Then,*

$$\begin{aligned} \left\|\nabla_q \phi[f]\right\|_2^2 &\leq 8\pi\varkappa(f)\mathcal{E}(f)\,, && (49)\\ \iint \sqrt{1+|p|^2} f(p,q)\mathrm{d}p\mathrm{d}q &\leq (1+\varkappa(f))\mathcal{E}(f)\,. && (50) \end{aligned}$$

Proof of Corollary 4.2. The bound (49) on $\|\nabla_q \phi\|_2$ is just a restatement of (47) in Proposition 4.1.

Noting that for ϕ given by (46) the energy functional (9) can be rewritten as

$$\mathcal{E}(f) = \iint \sqrt{1+|p|^2} f(p,q) \mathrm{d}p\, \mathrm{d}q - \frac{1}{8\pi} \|\nabla_q \phi[f]\|_2^2, \tag{51}$$

we see right away that (49) now implies (50). Q.E.D.

To state our next corollary we need the following lemma, which does not explicitly require f to be subcritical.

Lemma 4.3. *Assume that $f \in \mathfrak{P}_1 \cap \mathfrak{L}^\alpha$ for some $\alpha \geq 1$ (with $\alpha = \infty$ allowed). Then there exists a $C(\alpha)$ that depends only on α, such that the relative density of particles (given in (5)) satisfies the bound*

$$\|\rho\|_\gamma \leq C(\alpha) \|f\|_\alpha^\eta \mathcal{E}_p(f)^{1-\eta}, \tag{52}$$

where (we recall that) $\mathcal{E}_p(f)$ is the kinetic plus rest energy of f (i.e., l.h.s.(50)), and

$$\gamma := \frac{4\alpha - 3}{3\alpha - 2}, \qquad \eta := \frac{\alpha}{4\alpha - 3} \tag{53}$$

Remark 4.4. *Note that η in (53) is a decreasing function of α, taking values in $[1/4, 1]$, while γ in (53) is an increasing function of α, taking values in $[1, 4/3]$. So the optimal possible control of ρ is with exponent $\gamma = 4/3$, obtained when $\alpha = \infty$, while any $\alpha < \infty$ necessarily entails a weaker control on ρ. In particular, when $\alpha = 1$ nothing new is learned beyond what the definition of ρ says already.*

Proof of Lemma 4.3. Inspection of the proof, in [Hor81] and [GlSc85], of the corresponding $\|\rho\|_{4/3}$ bound for the relativistic Vlasov–Maxwell equations when $f \in \mathfrak{P}_1 \cap \mathfrak{L}^\infty$ is assumed shows that their proof generalizes to $f \in \mathfrak{P}_1 \cap \mathfrak{L}^\alpha$, all α. Thus,

$$\begin{aligned} \rho(q) &= \int_{|p|<P} f \mathrm{d}p + \int_{|p|\geq P} f \mathrm{d}p \\ &\leq \left(\int |f|^\alpha \mathrm{d}p \right)^{1/\alpha} \left(\tfrac{4\pi}{3} P^3\right)^{1/\alpha'} + \frac{1}{P} \int \sqrt{1+|p|^2} f \mathrm{d}p \\ &:= \left(\tfrac{4\pi}{3}\right)^{1/\alpha'} F(f) P^{3/\alpha'} + G(f) P^{-1} \end{aligned}$$

which upon optimizing in P yields

$$\rho(q) \leq C(\alpha)(F(f))^{\eta}(G(f))^{1-\eta}, \qquad \eta := \alpha/(4\alpha-3). \tag{54}$$

Raising both sides to power γ, integrating in q and applying Hölder's inequality with exponent $\delta := 3\alpha - 2 = \alpha/(\gamma\eta)$, noting that $\gamma(1-\eta)\delta' = 1$, yields

$$\begin{aligned}
\|\rho\|_\gamma^\gamma &\leq C(\alpha)^\gamma \int (F(f))^{\eta\gamma}(G(f))^{(1-\eta)\gamma}\,\mathrm{d}q \\
&\leq C(\alpha)^\gamma \Big(\int (F(f))^{\eta\gamma\delta}\mathrm{d}q\Big)^{1/\delta}\Big(\int (G(f))^{(1-\eta)\gamma\delta'}\mathrm{d}q\Big)^{1/\delta'} \\
&= C(\alpha)^\gamma \|f\|_\alpha^{\alpha/\delta}\Big(\iint \sqrt{1+|p|^2}f(p,q)\mathrm{d}p\,\mathrm{d}q\Big)^{\gamma(1-\eta)}
\end{aligned}$$

which gives the desired bound by virtue of the hypotheses of Lemma 4.3. Q.E.D.

As a spin-off of Corollary 4.2 and Lemma 4.3 we now have

Corollary 4.5. *Let $f \in \mathfrak{P}_1 \cap \mathfrak{L}^\alpha$ for some $\alpha \geq 3/2$. Assume furthermore that $\|f\|_{3/2} < \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$. Then there exists a $C(\alpha)$ that depends only on α, such that the relative density of particles satisfies the bound*

$$\|\rho\|_\gamma \leq C(\alpha)\|f\|_\alpha^\eta(1+\varkappa(f))^{1-\eta}\mathcal{E}(f)^{1-\eta}, \tag{55}$$

with γ and η related to α by (53).

Proof of Corollary 4.5. Under the hypotheses of Corollary 4.5 we can apply Corollary 4.2 which asserts in (50) that $\mathcal{E}_p(f) \leq (1+\varkappa(f))\mathcal{E}(f)$, which bound gives the desired bound on $\|\rho\|_\gamma$ by virtue of (52). Q.E.D.

Remark 4.6. *Note that in (55) γ now takes a value in $[6/5, 4/3]$, while η takes a value in $[1/4, 1/2]$. For the smallest possible $\alpha = 3/2$, (55) yields only an $\mathfrak{L}^{6/5}$ estimate of ρ, viz.*[7]

$$\|\rho\|_{6/5} \leq C\|f\|_{3/2}^{1/2}(1+\varkappa(f))^{1/2}\mathcal{E}(f)^{1/2}, \tag{56}$$

where C is some numerical constant independent of f. Of course, even weaker $\mathfrak{L}^{\gamma'}$ estimates of ρ hold, since under the hypotheses of Corollary 4.5, (55) remains true if in (55) α is replaced by any $\alpha' \in [1, \alpha)$, with $\gamma \mapsto \gamma'$ and $\eta \mapsto \eta'$ correspondingly.

[7] Incidentally, for $\vartheta \in (0,2)$ and $\delta \in [5/(6+2\vartheta), 2/(2+\vartheta))$, our $\hat{f}^{\psi_\delta} \propto (|p| - e^{-|q|}/|q|^\delta)_-^\vartheta \in \mathfrak{L}^{3/2}$ but $\hat{\rho}^{\psi_\delta} \notin \mathfrak{L}^{6/5}$. Yet we also have $\mathcal{E}_p^u(\hat{f}^{\psi_\delta}) = \infty$ for all $\delta \in [5/(6+2\vartheta), 2/(2+\vartheta)]$ when $\vartheta \in (0,2]$.

5 Subcritical solutions

We note that the a-priori bounds on the kinetic and potential energy functionals and on the $\mathfrak{L}^\gamma$ norm of ρ for subcritical f depend on f only through functionals which are conserved by sufficiently integrable classical solutions of the Vlasov evolution. This implies at once uniform bounds w.r.t. time on the corresponding quantities for sufficiently integrable classical solutions.

Corollary 5.1. *Let $t \mapsto f_t \in \mathfrak{P}_1 \cap \mathfrak{C}^1$, be a classical solution of rVP$^-$ over some time interval $t \in [0,T)$, and assume that initially and hence for all $t \in [0,T)$ we have $\|f_t\|_{3/2} < \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$. Then, uniformly in t, we have*

$$\left\|\nabla_q \phi_t\right\|_2^2 \leq 8\pi\varkappa(f_0)\mathcal{E}(f_0)\,, \tag{57}$$

$$\iint \sqrt{1+|p|^2} f_t(p,q)\mathrm{d}p\mathrm{d}q \leq (1+\varkappa(f_0))\mathcal{E}(f_0)\,, \tag{58}$$

and, for all $\alpha \in [1,\infty]$,

$$\|\rho_t\|_\gamma \leq C(\alpha)\|f_0\|_\alpha^\eta(1+\varkappa(f_0))^{1-\eta}\mathcal{E}(f_0)^{1-\eta} \tag{59}$$

with γ and η given by (53).

Proof of Corollary 5.1. Under the hypotheses of Corollary 5.1, which imply the conservation of energy, $\mathcal{E}(f_t) = \mathcal{E}(f_0)$, and of the $\mathfrak{L}^\alpha$ norms, $\|f_t\|_\alpha = \|f_0\|_\alpha$ (note that any function $f \in \mathfrak{P}_1 \cap \mathfrak{C}^1$ is automatically in all $\mathfrak{L}^\alpha$), the uniform bounds (57)–(58) follow at once from Corollary 4.2, and (59) from Corollary 4.5. Q.E.D.

Theorem 5.2. *Let $t \mapsto f_t \in \mathfrak{P}_1 \cap \mathfrak{C}^1$ be a classical solution of rVP$^-$ over some interval $t \in [0,T)$, launched by spherically symmetric data f_0 with compact support in p-space, vanishing for $p \times q = 0$, and assume that initially and hence for all $t \in [0,T)$ we have $\|f_t\|_{3/2} < \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$. Then the momentum support is uniformly bounded on $[0,T)$ and hence rVP$^-$ possesses a global classical solution.*

The proof of the above, which mimics closely the argument for global existence given in [GlSc85], hinges on the following estimate:

Lemma 5.3. *Let $t \mapsto f_t$ be a classical solution of rVP$^-$ over some interval $t \in [0,T)$. Then for any pair of exponents $\gamma < 3 < \alpha$ there exists a constant $C_{\alpha,\gamma}$ such that*

$$|\nabla_q \phi_t| \leq C_{\alpha,\gamma}\|f_0\|_\alpha^\theta\|\rho_t\|_\gamma^{1-\theta}P^\xi(t) \tag{60}$$

where

$$P(t) := \sup \left\{ |p| \big| (p,q) \in \operatorname{supp}(f_s),\ 0 \le s \le t \right\} \tag{61}$$

and

$$\theta := \frac{1-\gamma/3}{1-\gamma/\alpha} \in (0,1), \qquad \xi := 3(1-1/\alpha)\theta \tag{62}$$

Proof of Lemma 5.3. We have that $\phi_t = -|\mathrm{id}\,|^{-1} * \rho_t$ and hence

$$\begin{aligned} |\nabla_q \phi_t(q)| &\le \int_{|q-q'|<R} \frac{\rho_t(q')}{|q-q'|^2} \mathrm{d}q' + \int_{|q-q'|\ge R} \frac{\rho_t(q')}{|q-q'|^2} \mathrm{d}q' \\ &\le c_1'(\alpha)\|\rho_t\|_\alpha R^{1-3/\alpha} + c_2(\gamma)\|\rho_t\|_\gamma R^{1-3/\gamma} \\ &\le c_1(\alpha)\|f_t\|_\alpha P(t)^{3-3/\alpha} R^{1-3/\alpha} + c_2(\gamma)\|\rho_t\|_\gamma R^{1-3/\gamma}, \end{aligned}$$

which upon optimizing in R gives the desired result. Q.E.D.

Proof of Theorem 5.2: By spherical symmetry, $f_t(p,q) = \overline{f}_t(|p|,|q|,p\cdot q)$ and thus $\rho_t(q) = \overline{\rho}_t(|q|)$ so that we have $\nabla_q \phi_t(q) = M(|q|,t)|q|^{-3}q$, where

$$M(|q|,t) := 4\pi \int_0^{|q|} \overline{\rho}_t(r) r^2 \mathrm{d}r. \tag{63}$$

Note that $\lim_{|q|\to\infty} M(|q|,t) = 1$. Let the exponent $\alpha > 3$ be fixed, and let $\gamma = \frac{4\alpha-3}{3\alpha-2}$. Thus $9/7 < \gamma \le 4/3$. From Lemma 5.3 and Corollary 5.1 we then have that in the spherically symmetric case

$$|\nabla_q \phi_t(q)| \le \min\{|q|^{-2}, CP^\xi(t)\} \le 4(C^{-1/2}(P(t))^{-\xi/2} + |q|)^{-2} \tag{64}$$

where the constant C depends only on the initial data f_0 (specifically on its energy, its $\mathfrak{L}^{3/2}$ norm, and its $\mathfrak{L}^\alpha$ norm).

The next step, following the Glassey-Schaeffer argument, is to analyze the characteristics of the Vlasov equation in the spherically symmetric case, to conclude that, for any $T_0 \in [0,T)$,

$$P(T_0) \le P(0) + \sqrt{1+P^2(0)} + C^{1/2} P(T_0)^{\xi/2}\,. \tag{65}$$

We now compute:

$$\xi = 3\left(1-\frac{1}{\alpha}\right)\frac{1-\gamma/3}{1-\gamma/\alpha} = \frac{5\alpha-3}{3\alpha-3}\,. \tag{66}$$

Since $\xi < 2$ for $\alpha > 3$, this implies a uniform bound on $P(T_0)$ regardless of the size of C, and since T_0 was arbitrary in $[0,T)$, the theorem follows. Q.E.D.

6 Supercritical solutions

Recall that supercritical solutions are those for which $\|f_t\|_{3/2} > \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$. Notice that the energy of a supercritical solution can actually be strictly positive, zero, or strictly negative, and arbitrarily negative at that. While not much seems to be known about supercritical solutions with positive energy, *if* the energy of a solution is non-positive, i.e. if $\mathcal{E}(f_t) \leq 0$ (which implies that the solution is necessarily supercritical by Proposition 3.3), then one can rule out certain classes of global solutions.

In particular, when $\mathcal{E}(f_0) \leq 0$ we can rule out that f_t is stationary. This is an immediate consequence of the stationary virial theorem, Corollary 2.4, which asserts that $\mathcal{E}(f) > 0$ for any stationary solution. Since stationary solutions are global, we thus have ruled out a whole subclass of global solutions when $\mathcal{E}(f_0) \leq 0$. Note that the arguments just presented work without any symmetry assumption on f.

The dynamical virial theorem can be used to rule out further types of global solutions f_t with $\mathcal{E}(f_0) \leq 0$. By Theorem III of Glassey and Schaeffer [GlSc85], for $\mathcal{E}(f_0) < 0$ no spherical solution with compact support is global. We now prove the following generalization of the blow-up result in [GlSc85], which states that also some data with $\mathcal{E}(f_0) = 0$ lead to finite time blow-up.

Theorem 6.1. *Let $t \mapsto f_t \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ be a spherically symmetric supercritical classical solution of rVP$^-$ over some interval $[0, T)$, and assume that $\mathcal{E}(f_0) \leq 0$. In case that $\mathcal{E}(f_0) = 0$, also assume that $\mathcal{V}(f_0) \leq -1/2$. Then $T < \infty$.*

Proof of Theorem 6.1. For $\mathcal{E}(f_0) < 0$ this was proved (for compactly supported f_0) in [GlSc85]. Inspection of their proof of their Theorem III shows that it applies verbatim here, where compactness is replaced by the $\mathfrak{P}_3$ assumption.

In case that $\mathcal{E}(f_0) = 0$ we argue as follows. By (13) we find that $\mathcal{V}(f_t) < \mathcal{V}(f_0)$ for all $t \in [0, T)$, and since by hypothesis $\mathcal{V}(f_0) \leq -1/2$, we conclude that $\mathcal{V}(f_t) < -1/2$ for all $t \in [0, T)$; in fact, even $\mathcal{V}(f_t) < -C < -1/2$ for all $t \in (\epsilon, T)$ (Note that C will depend on ϵ and on f_0, but this is irrelevant for the argument). Next, as estimated in [GlSc85] in their proof of Thm. III, sphericity implies that $\left|\iint |q|^2 v \cdot \nabla_q \phi_t(q) f_t(p, q) \mathrm{d}p \, \mathrm{d}q\right| \leq 1$. Thus, by integrating (16) over t from 0 to T, and then using the above estimates, we find that

$$\iint |q|^2 \sqrt{1 + |p|^2} f_T(p, q) \mathrm{d}p \mathrm{d}q \leq \iint |q|^2 \sqrt{1 + |p|^2} f_0(p, q) \mathrm{d}p \mathrm{d}q + 2C\epsilon - (2C - 1)T, \quad (67)$$

and since $2C - 1 > 0$, it follows that the r.h.s. of (67) < 0 when T is large enough, while the l.h.s. is strictly positive. Therefore, T cannot be too large. Q.E.D.

Remark 6.2. *We don't know whether the condition $\mathcal{V}(f_0) \leq -1/2$ is actually sharp.*

Remark 6.3. *Classical solutions of nonrelativistic VP$^-$ don't blow up [Pfa92, Sch91].*

7 Critical solutions

Our global existence proof of classical solutions to the Cauchy problem for subcritical data relies heavily on the a-priori estimates for subcritical solutions obtained from a-priori bounds on the individual energy functionals $\mathcal{E}_p(f)$ and $\mathcal{E}_q(f)$. When f is critical so that $\|f_t\|_{3/2} = \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$, then our Proposition 3.3 still guarantees that $\inf \mathcal{E}(f_t) = 0$ and further that $\inf \mathcal{E}(f) \neq \min \mathcal{E}(f)$ over the set of critical data, but these estimates do not seem to produce a-priori bounds on the p-space and q-space energy functionals $\mathcal{E}_p(f)$ and $\mathcal{E}_q(f)$. So no global existence and uniqueness proof for critical classical solutions seems in sight. Curiously, no finite-time blow-up result is available either.

8 Some further interesting open problems

In addition to the wide open questions about critical solutions, there are a number of interesting problems that we would like to draw attention to.

Some of the assumptions in our global existence and uniqueness Theorem 1.1 are adapted from [GlSc85], in particular the compact p-support condition and the condition that f_0 vanish for $p \times q = 0$. We expect that the compact support condition can be replaced by a weaker one, like sufficiently rapid decay "at infinity," and that $f_0 = 0$ for $p \times q = 0$ can be dropped.

Since all solutions for which global existence and uniqueness has been proved are subcritical, one may want to consider the reverse question, whether a unique spherically symmetric global classical solution necessarily has to be subcritical. We suspect that the answer to this reverse problem is negative, but the problem is open. If the answer to this reverse problem is negative, it may be forthcoming more readily by studying the special subset of global-in-time solutions furnished by the stationary solutions to rVP$^-$, of which there are many examples (see [Bat89, HaRe07]).

Then there is the open question of the sharp values of C_β when $\beta > 3/2$.

Another interesting question is that of the weakest possible $\mathfrak{L}^\alpha$ norm for classical data which controls $\nabla\phi$. Our global existence and uniqueness proof for Theorem 1.1 invokes $f_0 \in \mathfrak{L}^\alpha$ for any $\alpha > 3$, beside subcriticality. We suspect that $\alpha = 3$

may be the critical α value for this question, indeed, but it would be good to have a definitive answer. A variant of this question should become particularly relevant when one asks for weaker solutions than classical, as e.g. in [dPLi89].

Finally, as pointed out earlier, our non-existence result for stationary solutions with $\mathcal{E}(f) \leq 0$ does not assume spherical symmetry. So one wonders how much of our other results generalizes to non-spherical solutions. This may seem primarily of mathematical interest, for without the sphericity assumption we are in danger of leaving the realm of physical validity of rVP$^-$. Yet, since spherical symmetry is never a perfect symmetry of nature, it is important to show that significant qualitative results do not sensitively depend on having exact spherical symmetry.

Note added After our paper was accepted for publication we discovered that our open question of the sharp values of C_β for $\beta > 3/2$ is implicitly answered by Proposition 1.1 of [LMR08a] which characterizes a subset of the compact Lane-Emden polytropes as optimizers in the variational principle (1.14) of [LMR08a], to which our variational principle (43) in Proposition 3.3, when restricted to $\beta > 3/2$, is equivalent. The quantitative evaluation of C_β, which requires numerical integration of the Lane-Emden polytrope equation, was meanwhile carried out by Brent Young and will be reported on elsewhere.

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APPENDIX

Derivation of rVP$^-$ with spherical symmetry from rVM

We begin by recalling the relativistic Vlasov–Maxwell equations for a two species plasma, containing a specie of N^+ positively, and another one of N^- negatively charged particles. For simplicity, it is assumed that all particles carry the same *magnitude* of charge and the same mass, as in an electron-positron plasma, and that there is overall an even number of particles so that $N^- + N^+ = 2N$. Units are chosen such that mass and magnitude of charge both $= 1$. Then, the particle density

functions $f_t^{\pm}(p,q) \in \mathfrak{P}_1 \cap \mathfrak{C}^1$ satisfy

$$\partial_t f_t^{\pm}(p,q) + v \cdot \nabla_q f_t^{\pm}(p,q) \pm \Big(E_t(q) + v \times B_t(q)\Big) \cdot \nabla_p f_t^{\pm}(p,q) = 0\,, \tag{68}$$

where velocity v and momentum p are related by Einstein's formula (2), while the electric field $E_t(q) \in (\mathfrak{L}^2 \cap \mathfrak{C}^1)(\mathrm{d}q)$ and the magnetic field $B_t(q) \in (\mathfrak{L}^2 \cap \mathfrak{C}^1)(\mathrm{d}q)$ at the space point $q \in \mathbb{R}^3$ at time $t \in \mathbb{R}$ satisfy the evolution equations

$$\partial_t B_t(q) = -\nabla_q \times E_t(q)\,, \tag{69}$$

$$\partial_t E_t(q) = \nabla_q \times B_t(q) - 4\pi \int_{\mathbb{R}^3} v(\nu^+ f_t^+ - \nu^- f_t^-)(p,q)\mathrm{d}p\,, \tag{70}$$

supplemented by the constraint equations

$$\nabla_q \cdot B_t(q) = 0\,, \tag{71}$$

$$\nabla_q \cdot E_t(q) = 4\pi \int_{\mathbb{R}^3} (\nu^+ f_t^+ - \nu^- f_t^-)(p,q)\mathrm{d}p\,. \tag{72}$$

Here, $\nu^{\pm}$ are the relative numbers of charges in the positive and negative specie; i.e.,[8] $N^- = \nu^- 2N$ and $N^+ = \nu^+ 2N$. Global well-posedness of the Cauchy problem for small initial data is known, and for large data under the additional assumption that no singularities occur near the light cone, see [GlSc88, GlSt86, GlSt87, KlSt02] and many further references therein. Note that the constraints propagate when satisfied by the initial data $E_0(q)$ and $B_0(q)$.

If we set *either* $\nu^+ = 0$ and $\nu^- = 1$ *or* $\nu^- = 0$ and $\nu^+ = 1$, and ignore the pertinent f^+, respectively f^- equation from the pair (68), the equations (68)–(72) of two-species rVM reduce to one-specie rVM. All spherically symmetric solutions of repulsive rVP (i.e. rVP$^+$, viz. $\sigma = +1$ in (1)–(4)) also solve this one-specie rVM. Indeed, one-specie rVM simply reduces to rVP$^+$ in the special situation of spherical symmetry, see [Hor90]. The reasons are: (i) all magnetic effects, in particular all electromagnetic waves, vanish in a spherically symmetric solution of rVM, rendering the charged particle interactions purely electric; and (ii) any two electric charges of the same sign repel each other electrically.

By contrast, spherically symmetric solutions of attractive rVP (i.e. rVP$^-$, viz. $\sigma = -1$ in (1)–(4)) do *not* solve rVM for any choice of ν^+ and ν^-, nor for any

[8] Note that in (68)–(72) we have chosen time and space scaled with $2N$ rather than N as in rVP.

other number of electrically charged particle species, so that any attempt to derive rVP$^-$ from rVM for two charged species of electrical particles would seem entirely misguided. However, we will now argue (convincingly, we hope) that certain families of *distributional* solutions of overall neutral two-species rVM (without spherical symmetry) converge to spherically symmetric solutions of rVP$^-$. Here is our argument.

Consider now an overall neutral two-species plasma, so $N^+ = N^- = N$, i.e. $\nu^+ = \nu^- = 1/2$. We are interested in the distributional solutions representing the *actual* empirical "densities" of the underlying $2N$-body system, which are sums of atomic measures (given below). To be able to work with such singular measures, we first need to regularize the equations of rVM by convolution with a smooth positive density function ϱ_ε with $SO(3)$ symmetry and compact support, satisfying $\int_{\mathbb{R}^3} \varrho_\varepsilon(q)\mathrm{d}^3q = 1$, which will be removed at the end of our construction.[9] The regularized rVM reads[10]

$$\partial_t \mu_t^\pm(p,q) + v\cdot\nabla_q \mu_t^\pm(p,q) \pm \Big((\varrho_\varepsilon * E_t)(q) + v\times(\varrho_\varepsilon * B_t)(q)\Big)\cdot\nabla_p \mu_t^\pm(p,q) = 0\,, \tag{73}$$

$$\begin{aligned}
\partial_t B_t(q) + \nabla_q \times E_t(q) &= 0\,, &(74)\\
-\partial_t E_t(q) + \nabla_q \times B_t(q) &= 2\pi \int_{\mathbb{R}^3} v(\varrho_\varepsilon * (\mu_t^+ - \mu_t^-))(p,q)\mathrm{d}p\,, &(75)\\
\nabla_q \cdot B_t(q) &= 0\,, &(76)\\
\nabla_q \cdot E_t(q) &= 2\pi \int_{\mathbb{R}^3} (\varrho_\varepsilon * (\mu_t^+ - \mu_t^-))(p,q)\mathrm{d}p\,. &(77)
\end{aligned}$$

Here we introduced the notation $\mu_t^\pm$ which could either mean density functions $f_t^\pm \in \mathfrak{P}_1 \cap \mathfrak{L}^1$ as before, or true measures $\mu_t^\pm \in \mathfrak{P}_1$. In particular, we can allow $\mu_0^\pm$ to be the *empirical relative "densities"* on $\mathbb{R}^6$ at time $t = 0$ of the underlying $2N$-body system, defined as follows. Letting particles with even index be positively charged and those with odd index negatively, we can identify any point $(p_{2k}, q_{2k})_{k=1}^N \in \mathbb{R}^{6N}_{even}$ (= N-positive-charges subspace of $2N$-body phase space) with a singular empirical relative "density" on $\mathbb{R}^6$,

$$\triangle^+(p,q) = \frac{1}{N}\sum_{k=1}^N \delta(p - p_{2k})\delta(q - q_{2k}), \tag{78}$$

[9]Note that this regularization breaks the Lorentz covariance; however, we do not invoke any Lorentz transformations in our reasoning, and for continuum solutions, when we let $\varrho_\varepsilon \to \delta$, we formally recover the Lorentz covariant relativistic Vlasov–Maxwell equations.

[10]The unfamiliar factors 2π in (75) and (77) are a consequence of $\nu^- = 1/2 = \nu^+$, which have been factored out from under the integrals and multiplied into 4π.

and each point $(p_{2k-1},q_{2k-1})_{k=1}^N \in \mathbb{R}^{6N}_{odd}$ with a singular empirical relative "density"

$$\triangle^-(p,q) = \frac{1}{N}\sum_{k=1}^{N}\delta(p-p_{2k-1})\delta(q-q_{2k-1})\,. \tag{79}$$

We write $(p_k(0),q_k(0))_{k=1}^{2N} \in \mathbb{R}^{12N}$ if the point in $2N$-body phase space is the initial phase point (at time $t=0$) of the phase space trajectory $t \mapsto (p_k(t),q_k(t))_{k=1}^{2N}$ of our plasma, and the corresponding empirical densities (dropping the quotes from now on) are denoted $\triangle_0^\pm(p,q)$, respectively $\triangle_t^\pm(p,q)$. Let the $\mu_0^\pm$ be given by some $\triangle_0^\pm(p,q)$. Then these initial empirical densities together with compatible initial data for the fields, $E_0(q)$ and $B_0(q)$, launch a subsequent evolution under (73)–(77) for which the $\mu_t^\pm$ are also given by empirical densities, viz. $\mu_t^\pm(p,q) = \triangle_t^\pm(p,q)$, characterized as follows.

Let the evolution equations for the dynamical variables of each particle, i.e. position $q_k(t)$ and linear momentum $p_k(t)$ be given by

$$\frac{\mathrm{d}q_k}{\mathrm{d}t}\bigg|_{q_k=q_k(t)} = \frac{p_k(t)}{\sqrt{1+|p_k(t)|^2}}\,, \tag{80}$$

$$\frac{\mathrm{d}p_k}{\mathrm{d}t}\bigg|_{p_k=p_k(t)} = e_k\left[(\varrho_\varepsilon * E_t)(q_k(t)) + \dot q_k(t)\times(\varrho_\varepsilon * B_t)(q_k(t))\right]\,, \tag{81}$$

with $e_k=-1$ if k is odd, and $e_k=+1$ if k is even. The above equations are the Einstein–Newton equations of motion, equipped with the Abraham–Lorentz expressions for the volume-averaged Lorentz force that acts on each particle. The evolution equations for the electric field $E_t(q)$ and the magnetic field $B_t(q)$ now read

$$\begin{aligned} \partial_t B_t(q) + \nabla_q \times E_t(q) &= 0\,, && (82)\\ -\partial_t E_t(q) + \nabla_q \times B_t(q) &= 2\pi\frac{1}{N}\sum_{k=1}^{2N} e_k\varrho_\varepsilon(q-q_k(t))\,\dot q_k(t)\,, && (83)\end{aligned}$$

and the constraint equations are

$$\begin{aligned} \nabla_q\cdot B(q,t) &= 0\,, && (84)\\ \nabla_q\cdot E(q,t) &= 2\pi\frac{1}{N}\sum_{k=1}^{2N} e_k\varrho_\varepsilon(q-q_k(t))\,, && (85)\end{aligned}$$

altogether known as the classical Maxwell–Lorentz field equations; note that (83) and (85) are just (75) and (77) with $\mu_t^\pm$ given by $\triangle_t^\pm$. It was proved recently in [BaDü01, KoSp00] that the dynamical equations (80)–(85) are globally well posed as Cauchy problem in convenient Hilbert spaces; see also [KuSp00, Spo04]. Let $t \mapsto (p_k(t), q_k(t))_{k=1}^{2N} \in \mathbb{R}^{12N}$ be the particle phase space trajectory of a global finite energy solution to this Abraham–Lorentz[11] model (80)–(85). Then the corresponding dynamical empirical densities $\triangle_t^\pm(p,q)$ satisfy regularized rVM (73)–(77) in the sense of distributions. Thus the Abraham–Lorentz model is entirely equivalent to the regularized rVM *restricted to empirical densities* $\triangle_t^\pm$.

We now note a very important point about the relationship of distributional solutions of (regularized) rVM and its continuum solutions. Recall that in the introduction we pointed out that f_t "should really be thought of as a continuum approximation to a *merely normalized* (i.e. relative) empirical "density" on (p,q)-space of an actual individual N-body system." In this sense, assume that N is sufficiently large so that for both species $\triangle_0^\pm \approx f_0^\pm \in \mathfrak{P}_1 \cap \mathfrak{L}^1$ closely in measure (i.e. w.r.t. some Kantorovich-Rubinstein distance). Then *on a suitably short time scale* the evolutions of the $\triangle_t^\pm$ under (73)–(77) will be reasonably closely approximated by solutions of (73)–(77) with the initial data $\triangle_0^\pm$ replaced by $f_0^\pm \in \mathfrak{P}_1 \cap \mathfrak{L}^1$, and with the initial data for the fields, E_0 and B_0, replaced accordingly.[12] On longer time scales, various deviations of the rVM evolution for regular initial data will become visible, and as we shall see now, under favorable conditions, one of those long-time evolutions is captured precisely by regularized rVP$^-$. For continuum solutions we can subsequently let $\varrho_\varepsilon \to \delta$, recovering rVP$^-$.

To see this, contemplate that both $\triangle_0^\pm \approx f_0^\pm \equiv f_0$, with f_0 spherical. Then on the conventional short Vlasov time scale we will just find non-interacting perfect gas dynamics, for $f_t^+ - f_t^- \equiv 0$, then, in (70) and (72) with $\nu^- = 1/2 = \nu^+$. However, if *all* the particles are i.i.d. by f_0, then on a longer time scale we should obtain rVP$^-$ for f_t, and here is why. First of all, by (approximate) spherical symmetry of the $2N$-body plus field system we expect that magnetic effects can again be neglected,

[11] These semi-relativistic equations of Abraham–Lorentz electrodynamics are a non-Lorentz covariant regularization of formally Lorentz covariant formal Lorentz electrodynamics with point charges (which has the unpleasant feature that its formal equations are mathematically ill-defined without regularization). The semi-relativistic Abraham–Lorentz model is chosen purely for the ease of the discussion. A fully Lorentz covariant regularized Lorentz model is available (see [ApKi01, ApKi02, Spo04]) but is considerably more complicated.

[12] For a rigorous proof of this for a scalar caricature of rVM, see [EKR09].

so we set $B \equiv 0$. But then (85), (82), and (83) are solved by Coulomb's formula

$$E_t(q) = -\frac{1}{2N}\sum_{k=1}^{2N} e_k \nabla_q \left(|\mathrm{id}\,|^{-1} * \varrho_\varepsilon(\,\cdot\, - q_k(t))\right)(q) \tag{86}$$

Evaluating r.h.s. (81) with this formula for E, setting $B \equiv 0$, we find for particle ℓ,

$$\begin{aligned} e_\ell(\varrho_\varepsilon * E_t)(q_\ell(t)) = &- \frac{e_\ell}{2N}\sum_{k=1}^{N}\left(\varrho_\varepsilon * \nabla_q \left(|\mathrm{id}\,|^{-1} * \varrho_\varepsilon(\,\cdot\, - q_{2k}(t))\right)\right)(q_\ell(t)) \\ &+ \frac{e_\ell}{2N}\sum_{k=1}^{N}\left(\varrho_\varepsilon * \nabla_q \left(|\mathrm{id}\,|^{-1} * \varrho_\varepsilon(\,\cdot\, - q_{2k-1}(t))\right)\right)(q_\ell(t)) \end{aligned} \tag{87}$$

where we made use of our convention that positive particles carry an even, negative an odd index. Now observe that since by hypothesis *all* particles are i.i.d. by f_0, all *but one* of the force terms in (87) are i.i.d. random variables, the exception being

$$\left(\varrho_\varepsilon * \nabla_q \left(|\mathrm{id}\,|^{-1} * \varrho_\varepsilon(\,\cdot\, - q_\ell(t))\right)\right)(q_\ell(t)) \equiv 0, \tag{88}$$

which states that the Coulomb self-force on the particle vanishes. But then, if ℓ is even, a term from the first sum is missing, and if ℓ is odd, a term from the second sum is missing. For all the other terms, since *all* particles are i.i.d., we can use that both $\triangle_t^{\pm} \approx f_t$, and paying attention to the correct normalization,[13] we now find that

$$e_\ell(\varrho_\varepsilon * E_t)(q_\ell(t)) \approx \frac{1}{2N}\left(\varrho_\varepsilon * \nabla_q \left(|\mathrm{id}\,|^{-1} * (\varrho_\varepsilon * \rho_t)\right)\right)(q_\ell(t)) \tag{89}$$

which is independent of e_ℓ: this means that each particle is acted on by a net attractive central force, for from each particle's perspective the rest of the system is singly oppositely charged, as the rest of the system always contains one more of the oppositely charged than the equally charged particles; the force is (approximately) central by (approximate) spherical symmetry. We therefore introduce

$$\phi_t^\epsilon := -\frac{1}{2N}\varrho_\varepsilon * \left(|\mathrm{id}\,|^{-1} * (\varrho_\varepsilon * \rho_t)\right) \tag{90}$$

[13] Mathematically this is a nice instance where one might be tempted to set two extremely huge numbers N and $N-1$ equal, but here their difference matters, and $N-(N-1) = 1 \neq 0 = N-N$.

replace $e_\ell E_t(q_\ell(t))$ by $-\nabla_q \phi_t^\epsilon(q_\ell(t))$ for both positive and negative charges in the regularized rVM (noting that ∇_q and $\varrho_\varepsilon *$ commute) and also replace $\triangle_t^\pm$ by f_t, upon which both equations (73) reduce to the same regularized Vlasov equation

$$\Big(\partial_t + v \cdot \nabla_q - \nabla_q \phi_t^\epsilon(q) \cdot \nabla_p\Big) f_t(p,q) = 0, \tag{91}$$

in which the velocity $v \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$ of a (point) particle of unit mass are again related by Einstein's formula (2), and where ϕ_t^ϵ is given by the r.h.s. of (90). Note that (91) is decoupled from the Maxwell-Lorentz field equations. If f_t is sufficiently regular, and uniformly so for all ϵ, we may now let $\varrho_\varepsilon \to \delta$ and find that the resulting f_t is a solution to the Vlasov equation

$$\Big(\partial_t + v \cdot \nabla_q - \nabla_q \phi_t(q) \cdot \nabla_p\Big) f_t(p,q) = 0, \tag{92}$$

where now

$$\phi_t(q) = -\frac{1}{2N} \left(|\mathrm{id}\,|^{-1} * \rho_t\right)(q). \tag{93}$$

Clearly, the scalar field ϕ_t satisfies the Poisson equation

$$\Delta_q \phi_t(q) = 4\pi \frac{1}{2N} \int_{\mathbb{R}^3} f_t(p,q)\mathrm{d}p \tag{94}$$

with asymptotic condition

$$\phi_t(q) \asymp -(2N|q|)^{-1} \tag{95}$$

when $|q| \to \infty$. By a final rescaling of space and time variables we can get rid of the factor $1/2N$ and thus have obtained rVP$^-$.

Remark 8.1. *The i.i.d. assumption on* all *particles is very important. Indeed, even if we merely assume that the particles of each specie are i.i.d. w.r.t. f_0 separately, then we can still have that particles of opposite species are strictly correlated, viz. wherever a positive particle is located, a negative one is, too. This is the classical analog of a neutral gas of "positronium atoms," and no electric rVP$^-$ will result.*

References

[Aly89] Aly, J.J., *On the lowest energy state of a collisionless self-gravitating system under phase space volume constraints*, Mon. Not. R. astr. Soc. **241** (1989), 15–27.

[ApKi01] Appel, W., and Kiessling, M.K.-H., *Mass and spin renormalization in Lorentz Electrodynamics*, Ann. Phys. (N.Y.) **289** (2001), 24–83.

[ApKi02] Appel, W., and Kiessling, M.K.-H., *Scattering and Radiation damping in Gyroscopic Lorentz Electrodynamics*, Lett. Math. Phys. **60** (2002), 31–46.

[Bat89] Batt, J., *Steady state solutions of the relativistic Vlasov–Poisson system*, pp. 1235–1247 in "Proceedings of the Fifth Marcel Grossmann Meeting on General Relativity" (Perth, 1988), World Scientific, Teaneck, NJ (1989).

[BaDü01] Bauer, G., and Dürr, D., *The Maxwell–Lorentz system of a rigid charge,* Ann. Inst. H. Poincaré **2** (2001), 179-196.

[dPLi89] DiPerna, R.J., and Lions, P.L., *Global Weak Solutions of Vlasov–Maxwell systems*, Commun. Pure Appl. Math. **42** (1989), 729–757.

[EKR09] Elskens, Y., Kiessling, M.K.-H., and Ricci, V. *The Vlasov limit for a system of particles which interact with a wave field*, Commun. Math. Phys. **285** (2009), 673–712.

[GlSc85] Glassey, R., and Schaeffer, J., *On symmetric solutions to the relativistic Vlasov–Poisson system*, Commun. Math. Phys. **101** (1985), 459-473.

[GlSc88] Glassey, R., and Schaeffer, J., *Global existence for the relativistic Vlasov–Maxwell system with nearly neutral initial data*, Commun. Math. Phys. **119** (1988), 353-384.

[GlSc01] Glassey, R., and Schaeffer, J., *On global symmetric solutions to the relativistic Vlasov-Poisson equation in three dimensions*, Math. Meth. Appl. Sci. **24** (2001), 143-157.

[GlSt86] Glassey, R., and Strauss, W., *Singularity formation in a collisionless plasma could occur only at high velocities*, Arch. Rat. Mech. Analysis **92** (1986), 59-90.

[GlSt87] Glassey, R., and Strauss, W., *Absence of shocks in an initially dilute collisionless plasma*, Commun. Math. Phys. **113** (1987), 191-208.

[HaRe07] Hadžić, M., and Rein, G., *Global existence and nonlinear stability for the relativistic Vlasov–Poisson system in the gravitational case*, Indiana Univ. Math. J. **56** (2007), 2453-2488.

[Hor81] Horst, E., *On the classical solutions of the initial valum problem of the unmodified Vlasov equations. I& II*, Math. Meth. Appl. Sci. **3** (1981), 229–248; **4** (1982), 19–23.

[Hor90] Horst, E., *Symmetric plasmas and their decay*, Commun. Math. Phys **126** (1990), 613–633.

[KlSt02] Klainerman, S., and Staffilani, G., *A new approach to study the Vlasov–Maxwell system*, Commun. Pure Appl. Anal. **1** (2002), 103–125.

[KoSp00] Komech, A., and Spohn, H., *Long-time asymptotics for the coupled Maxwell–Lorentz Equations*, Commun. PDE **25** (2000), 559–584.

[KuSp00] Kunze, M., and Spohn, H., *Slow motion of charges interacting through the Maxwell field*, Commun. Math. Phys. **212** (2000), 437–467.

[LMR08a] Lemou, M., Méhats, F., and Raphaël, P., *Structure of the linearized gravitational Vlasov–Poisson system close to a polytropic ground state*, SIAM J. Math. Anal. **39** (2008), 1711–1739.

[LMR08b] Lemou, M., Méhats, F., and Raphaël, P., *Stable self-similar blow-up dynamics for the three-dimensional relativistic Vlasov–Poisson system*, J. Amer. Math. Soc. **21** (2008), 1019–1063.

[Lie83] Lieb, E.H., *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities*, Ann. Math. **118** (1983), 349–374.

[Pfa92] Pfaffelmoser, K., *Global classical solutions of the Vlasov–Poisson system in three dimensions with generic initial data*, J. Diff. Eq. **95** (1992), 281-303.

[Rei97] Rein, G., *Self-gravitating systems in Newtonian theory – the Vlasov–Poisson system*, in: *Mathematics of gravitation*, Part I, Banach Center Publ. **41**, pp. 179–194 Warzawa (1997).

[Ren94] Rendall, A., *The Newtonian limit for asymptotically flat solutions of the Vlasov–Einstein system*, Commun. Math. Phys. **163** (1994), 89–112.

[Sch91] Schaeffer, J., *Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions*, Commun. PDE **16** (1991), 1313-1335.

[Spo04] Spohn, H., *Dynamics of Charged Particles and Their Radiation Fields*, Cambridge Univ. Press, Cambridge (2004).

[Tal76] Talenti, B., *Best constant in Sobolev inequality*, Ann. Matem. Pura Appl. **110** (1976), 353–372.

[WZS88] Wiechen, H., Ziegler, H.J., and Schindler, K., *Collisionless relaxation of self-gravitating matter: the lowest energy state*, Mon. Not. R. astr. Soc. **232** (1988), 623–646.